Final Exam for Complex Analysis **09/04/2025**

Student number: **Name:**

Please write your name and student number on top of each sheet that you hand in.

You get 10 points for participating. *Good luck!*

1. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(x+iy) = (x + e^x \cos y) + i(e^x \sin y - y)$ for all $x, y \in \mathbb{R}$.
   (a) (10 pts) Determine the set of all points in $\mathbb{C}$ at which f is complex differentiable.
   (b) (10 pts) Compute the line integral $\int_L f(z)\,dz$ where L is the line segment from 0 to $i\pi$ in $\mathbb{C}$.
2. (15 pts) Let C denote the circle $|z| = 3$ oriented counterclockwise. Compute the integral
$$\int_C \frac{\sin(z^2)}{z^3(z-1)}\,dz.$$
3. (15 pts) Use Rouché's theorem to to determine the number of zeros, counting with multiplicities, of the polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ inside the disk $|z| < 2$.
4. (15 pts) Suppose f and g are entire functions and D is a compact subset of $\mathbb{C}$. Show that $|f(z)| + |g(z)|$, for $z \in D$ takes its maximum value on the boundary of D. (**Suggestion:** Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.)
5. (a) (10 pts) Let C_R denote the half circle $\{z : |z| = R,\ \mathrm{Im}(z) \geq 0\}$. Prove that
$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2+1}\,dz = 0$$
   (b) (15 pts) Using complex residues and part (a), compute the (real valued) integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx.$$